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TOTAL EDGE DOMINATION INTEGRITY IN GRAPHS

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Abstract. Domination concept has been used in many fields like Science, Technology, Engineering, Communication Networks,etc. Total edge domination is one of the well-known domination concepts. A set D of edges is called a total edge dominating set of G if every edge in G is adjacent to at least one edge in D. In this paper, the concept of total edge domination integrity is introduced as a new vulnerability measure and some properties, bounds and total edge domination integrity of some graph classes are examined. Total edge domination integrity of a graph G is denoted by $TDI^{'}(G)$ and defined as $TDI^{'}(G) = min\{|S| + m(G-S) : S \subseteq E(G)\}$ where S is a total edge dominating set of G and $m(G-S)$ is an order of maximum component of $G-S$. In this study, all the graphs considered simple, finite and undirected.

Keywords: Communication networks, vulnerability, total edge domination, edge integrity, total edge domination integrity.

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1 Introduction

Networks have become more important with the increasing demand for information transfer in real life at any type of area. A communication network consists of nodes and links that connect these nodes. The efficiency of the network is effected by any damages on the nodes or the links of the communication network. The stability of network is the most important concept after the damages that appear in the network. If the stability is mentioned, then the concept of vulnerability comes to mind.The resistance of a network after any failures on nodes or links in any type of communication network, is called vulnerability value.

A communication network can be modeled by a graph where nodes are represented by vertices and links by edges. Thus the vulnerability parameters on graph models are studied in vulnerability analysis of networks. There have been many parameters introduced for the measurement of vulnerability. Some of these are connectivity, edge connectivity, tenacity, toughness, integrity, edge integrity, domination integrity etc.

In analysis of the vulnerability of a communication network, the most known three fundamental questions are *(i)* the number of elements that must be removed to disconnect a network *(ii)* the number of elements that are not functioning and (iii) the size of the largest remaining component, in which mutual communication still exists.

Connectivity is a parameter that gives the answer to the first question. The connectivity of any graph G, written $\kappa(G)$, is the minimum order of a vertex set S such that $G-S$ is disconnected or has only one vertex. The edge connectivity of G, written $\lambda(G)$, is the minimum size of a disconnecting set (West, 2001).

Integrity and edge integrity are other well-known vulnerability parameters which try to find answers to second and third questions. Both types of integrity were introduced by Barefoot et al. (1987). Vertex-integrity is defined as

$$I(G) = min\{|S| + m(G-S) : S \subset V(G)\}$$

where $m(G-S)$ denotes the maximum order of a component of $G-S$ and edge integrity as

$$I^{'}(G) = \min\{|S| + m(G-S) \ : S \subseteq E(G)\} \ .$$

Failures on vertices or edges which have specific properties play a great role in the vulnerability analysis of communication networks. For example, domination is a well-known concept in network design and it has a wide range of applications to many areas like Science, Technology, Engineering.

Definition 1. *A subset X of E is called an edge dominating set of G if every edge not in X is adjacent to some edge in X (Mitchell & Hedetniemi, 1977; Arumugam & Velammal, 1998).*

Definition 2. *The edge domination number $\gamma'(G)$ (or γ' for short) of G is the minimum cardinality taken over all edge dominating sets of G (Arumugam & Velammal, 1998).*

Kılıç & Beşirik (2018) have introduced the concept of domination edge integrity as a new vulnerability parameter and it is defined as follows.

Definition 3. *The domination edge integrity of a connected graph G denoted by $DI^{'}(G)$ and defined by*

$$DI^{'}(G) = \min\{|S| + m(G-S) \ : S \text{ is an edge dominating set } \}$$

where $m(G-S)$ is the order of a maximum component of $G-S$ (Kılıç & Beşirik, 2018) .

Definition 4. *A subset S of $E(G)$ is a $DI^{'}$-set if $DI^{'}(G) = |S| + m(G-S)$ and S is an edge dominating set of G (Kılıç & Beşirik, 2018).*

2 Total Edge Domination Integrity

Domination is a well-known concept in network design and it has been used in many applications and in many areas of science. There are various types of domination defined by researchers in recent studies. Some of them are edge domination, connected domination, global domination, total edge domination. In this paper, total edge domination was considered.

Definition 5. *A set D of edges is a total edge dominating set of G if every edge in G is adjacent to at least one edge in D. The total edge domination number $\gamma_t^{'}(G)$ is defined to be the minimum number of edges in a total edge dominating set of G (Kulli & Patwari, 1991).*

Proposition 1. *For any connected graph G with $n \geq 4$ vertices and q edges, $\gamma_t^{'} \geq \frac{q}{\Delta'}$ (Kulli & Patwari, 1991).*

Proposition 2. *For $k \geq 1$,*

$$\gamma_t^{'}(P_n) = \begin{cases} \frac{n}{2}, & n \text{ is even} \\ \left\lceil \frac{n}{2} \right\rceil, & n = 4k+1 \\ \left\lceil \frac{n}{2} \right\rceil, & n = 4k+3 \end{cases}$$

(Kulli & Patwari, 1991).

Proposition 3. *For $k \geq 1$,*

$$\gamma_t^{'}(C_n) = \begin{cases} \frac{n}{2}, & n = 4k \\ \left\lceil \frac{n}{2} \right\rceil, & n = 4k+1 \quad \text{and } n = 4k+3 \\ \frac{n}{2}+1, & n = 4k+2 \end{cases}$$

(Kulli & Patwari, 1991).

Proposition 4. *For $2 \leq m \leq n$, $\gamma_t^{'}(K_{m,n}) = m$ (Kulli & Patwari, 1991).*

Proposition 5. *For $n \geq 4$, $\gamma_t^{'}(W_n) = \left\lfloor \frac{n}{2} \right\rfloor$ (Kulli & Patwari, 1991).*

Domination has been applied to many problem models such as monitoring communication or networks, location, routing, etc. Total edge domination plays a great role in such kind of many real life problems. For example, a transport network can be considered as a graph model.(Paths in a transport network can be represent by edges and stops can be represent by vertices). We assume that if one of the paths in the transport network is damaged, then transport can be continued with the other adjacent path. But if the adjacent path is also damaged, then the transport system may fail to function properly. Thus, when total edge dominating sets are removed in a network, the damage is vital. This motivated us to introduce total edge domination integrity as a new parameter of stability when total edge dominating sets are damaged.

Definition 6. *The total edge domination integrity of a graph G is*

$$TDI^{'}(G) = min\{|S| + m(G-S) : S \subseteq E(G)\}$$

where $m(G-S)$ denotes the order of a maximum component of $G-S$ and S is a total edge dominating set of G.

Definition 7. *A set $S \subseteq E(G)$ is a $TDI^{'}$-set if $TDI^{'}(G) = |S| + m(G-S)$ and S is a total edge dominating set of G.*

If two graphs have same edge connectivity, edge integrity and domination edge integrity values, then these parameters are not enough to distinguish them. In this case, a new parameter which will distinguish these graphs is needed. Otherwise how can a network designer determine which network is more stable than the other? Is the total edge domination integrity can be regarded as a vulnerability parameter that compare these graphs in resistance?

Let's see this with a simple comparison between two graphs. Assume that G_1 and G_2 are graphs with same number of vertices and edges as follows in Figure 1.

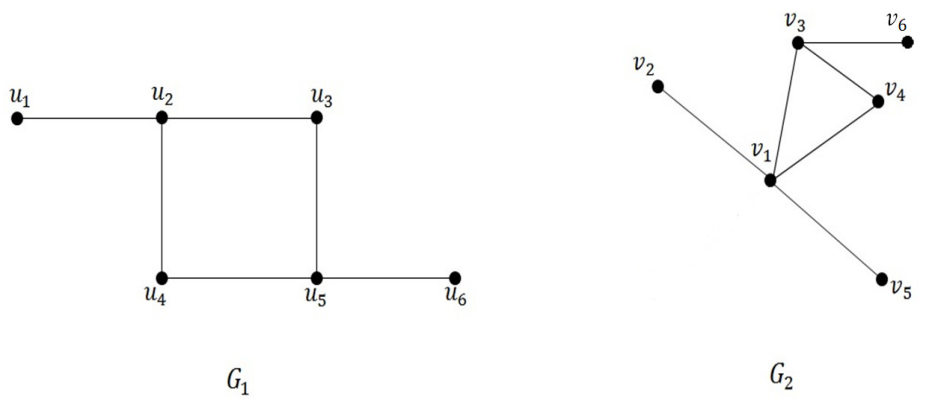

Figure 1: Graph G_1 and graph G_2

For G_1 and G_2, $\lambda(G_1) = \lambda(G_2) = 1$, $I^{'}(G_1) = I^{'}(G_2) = 5$ and $DI^{'}(G_1) = DI^{'}(G_2) = 5$. So, edge connectivity, edge integrity and domination edge integrity do not distinguish between G_1 and G_2. Total edge domination integrity values of G_1 and G_2 are computed as follows.

Consider $S_1 = \{< u_2u_3 >, < u_3u_5 >, < u_4u_5 >\}$ as a total edge dominating set of G_1 then $m(G_1 - S_1) = 3$. There does not exist a total dominating set of G_1 such that $|X| + m\,(G_1 - X) < |S_1| + m(G_1 - S_1)$. So, $TDI^{'}(G_1) = 3 + 3 = 6$.

Consider $S_2 = \{< v_1v_3 >, < v_1v_4 >\}$ as a total edge dominating set of G_2 then $m(G_2 - S_2) = 3$. There does not exist a total edge dominating set of G_2 such that $|X| + m\,(G_2 - X) < |S_2| + m(G_2 - S_2)$. So, $TDI^{'}(G_2) = 2 + 3 = 5$.

For G_1 and G_2, $TDI^{'}(G_1) = 6$ and $TDI^{'}(G_2) = 5$. So, it can be said that G_1 is more stable than G_2. Then, total edge domination integrity is a suitable measure of vulnerability which distinguishes between these graphs.

Observation 1. $3 \leq TDI^{'}\,(G) \leq |E| + 1$.

Proof. $TDI^{'}$ takes its minimum value in path graph P_3 since P_3 has two edges. To obtain components with minimum order in any connected graph G all edges must be removed from the graph. In other words total edge dominating set should contain all the edges of the graph. In this case we obtain isolated vertices thus $m\,(G - S)$ is 1 and minimum. Hence $TDI^{'}(G) \leq |E|+1$ and $|E| + 1$ is a maximum upper bound value for $TDI^{'}$. □

Observation 2. $I^{'}(G) \leq DI^{'}(G) \leq TDI^{'}(G)$.

Proof. Kulli & Patwari (1991) proved that $\gamma^{'}(G) \leq \gamma_t^{'}(G)$. So we can clearly see $DI^{'}(G) \leq TDI^{'}(G)$. Since $I^{'}(G) \leq DI^{'}(G)$ (Kılıç & Beşirik, 2018), then $I^{'}(G) \leq DI^{'}(G) \leq TDI^{'}(G)$. □

Observation 3. $I(G) \leq DI(G) \leq TDI(G) \leq TDI^{'}\,(G)$.

Proof. Beşirik (2019) proved that $I(G) \leq DI(G) \leq TDI(G)$. Since removing a vertex from the graph can remove many edges, it is clear that $TDI(G) \leq TDI^{'}\,(G)$. Hence, we obtain $I(G) \leq DI(G) \leq TDI(G) \leq TDI^{'}\,(G)$.

□

Theorem 1. *Let G be a graph of order $n \geq 3$. Then $TDI^{'}\,(G) \geq \gamma_t^{'} + 1$.*

Proof. Let G be a graph of order $n \geq 3$ and S is a $\gamma_t^{'}$-set of G. So, $|S| = \gamma_t^{'}$ and $m(G - S) \geq 1$. Then,

$$\begin{aligned} TDI^{'}\,(G) &= \min_{S \subseteq E(G)} \{|S| + m\,(G - S)\} \\ &\geq \min\{\gamma_t^{'} + 1\} \\ &\geq \gamma_t^{'} + 1 \end{aligned}$$

□

Theorem 2. *Let G be a connected graph of order $n \geq 4$ and q edges. Then, $TDI^{'}\,(G) \geq \frac{q}{\Delta^{'}} + 1$.*

Proof. Let G be a connected graph with $n \geq 4$ vertices, q edges and S is a $\gamma_t^{'}$-set of G. So, $|S| = \gamma_t^{'}$ and $m(G - S) \geq 1$. From *Theorem 1*, $TDI^{'}(G) \geq \gamma_t^{'} + 1$.
It has been proved that $\gamma_t^{'} \geq \frac{q}{\Delta^{'}}$ in *Proposition 1*. Since $\gamma_t^{'} \geq \frac{q}{\Delta^{'}}$, then we have $TDI^{'}\,(G) \geq \gamma_t^{'} + 1 \geq \frac{q}{\Delta^{'}} + 1$. □

Theorem 3. *For $k \geq 1$ and $n \geq 9$,*

$$TDI^{'}\,(P_n) = \begin{cases} \frac{n}{2} + 3, & n \text{ is even} \\ \left\lfloor \frac{n}{2} \right\rfloor + 3, & n = 4k + 1 \\ \left\lceil \frac{n}{2} \right\rceil + 3, & n = 4k + 3 \end{cases}$$

Proof. Let P_n be a path graph , $n \geq 9$ and $k \geq 1$. It has been proved that $\gamma_t^{'}(P_n) = \frac{n}{2}$ when n is even, $\gamma_t^{'}(P_n) = \lfloor \frac{n}{2} \rfloor$ when $n = 4k+1$, $\gamma_t^{'}(P_n) = \lceil \frac{n}{2} \rceil$ when $n = 4k+3$ in *Proposition 2*. For any minimum total edge dominating set S of P_n, $m(P_{\mathrm{n}} - S) = 3$. Therefore, $TDI^{'}(P_n) \leq \gamma_t^{'}(P_n) + 3$. If X is any total edge dominating set of P_{n}, then $|X| + m(P_n - X) \geq \gamma_t^{'}(P_n) + 3$. Hence the result is obtained. □

Theorem 4. *For $k \geq 1$ and $n \geq 7$,*

$$TDI^{'}(C_n) = \begin{cases} \frac{n}{2} + 3, & n = 4k \\ \lceil \frac{n}{2} \rceil + 3, & n = 4k+1 \text{ and } n = 4k+3 \\ \frac{n}{2} + 4, & n = 4k+2 \end{cases}$$

Proof. Let C_n be a cycle graph, $n \geq 7$ and $k \geq 1$. It has been proved that $\gamma_t^{'}(C_n) = \frac{n}{2}$ when $n = 4k$, $\gamma_t^{'}(C_n) = \lceil \frac{n}{2} \rceil$ when $n = 4k+1$ and $n = 4k+3$, $\gamma_t^{'}(C_n) = \frac{n}{2} + 1$ when $n = 4k+2$ in *Proposition 3*. For any minimum total edge dominating set S of C_n, $m(C_{\mathrm{n}} - S) = 3$. Therefore, $TDI^{'}(C_n) \leq \gamma_t^{'}(C_n) + 3$. If X is any total edge dominating set of C_{n}, then $|X| + m(C_n - X) \geq \gamma_t^{'}(C_n) + 3$. Hence the result is obtained. □

Theorem 5. *For $m \geq 2$ and $n \geq 2$, $TDI^{'}(K_{m,n}) = min\{m, n\} + m + n - 1$.*

Proof. Let $K_{m,n}$ be a complete bipartite graph, $m \geq 2$ and $n \geq 2$. From *Proposition 4*, we know that $\gamma_t^{'}(K_{m,n}) = min\{m, n\}$. For any minimum total edge dominating set S of $K_{m,n}$, $m(K_{\mathrm{m,n}} - S) = m + n - 1$. Therefore, $TDI^{'}(K_{m,n}) \leq \gamma_t^{'}(K_{m,n}) + m + n - 1$. If X is any total edge dominating set of $K_{m,n}$, then $|X| + m(K_{m,n} - X) \geq \gamma_t^{'}(K_{m,n}) + m + n - 1$. Hence the result is obtained.

□

Theorem 6. *For $m \geq 2$, $TDI^{'}(K_{1,m}) = m + 1$.*

Proof. Let $K_{1,m}$ be a star graph and $m \geq 2$. It is clear that $\gamma_t^{'}(K_{1,m}) = 2$. For any minimum total edge dominating set S of $K_{1,m}$, $m(K_{1,\mathrm{m}} - S) = m - 1$. Therefore, $TDI^{'}(K_{1,m}) \leq \gamma_t^{'}(K_{1,m}) + m - 1 = m + 1$. If X is any total edge dominating set of $K_{1,m}$, then $|X| + m(K_{1,m} - X) \geq m + 1$. Hence the result is obtained.

□

Theorem 7. *For $n \geq 4$, $TDI^{'}(W_n) = \lfloor \frac{n}{2} \rfloor + n$.*

Proof. Let W_n be a wheel graph and $n \geq 4$. From *Proposition 5*, we know that $\gamma_t^{'}(W_n) = \lfloor \frac{n}{2} \rfloor$. For any minimum total edge dominating set S of W_n, $m(W_{\mathrm{n}} - S) = n$. Therefore, $TDI^{'}(W_n) \leq \gamma_t^{'}(W_n) + n$. If X is any total edge dominating set of W_{n}, then $|X| + m(W_n - X) \geq \gamma_t^{'}(W_n) + n$. Hence the result is obtained. □

3 Conclusion

In this paper, total edge domination integrity is introduced as a new vulnerability parameter and total edge domination integrity of basic graph classes P_n, C_n, W_n, $K_{1,n}$, $K_{m,n}$ are examined. The bounds and some properties for total edge domination integrity value of a graph are determined. In future work, we plan to generalize results of total edge domination integrity value under basic graph operation classes.

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